Completeness of the WDS method in Checking Positivity of Integral Forms *

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Abstract: Examples show that integral forms can be efficiently proved positive semidefinite by the WDS method, but it was unknown that how many steps of substitutions are needed, or furthermore, which integral forms is this method applicable for. In this paper, we give upper bounds of step numbers of WDS required in proving that an integral form is positive definite, positive semidefinite, or not positive semidefinite, thus deducing that the WDS method is complete.

Key words: integral form; weighted difference substitution; positivity

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1 Introduction

Polynomials play key roles in many fields of the system theory [1], fundamental problems in automatic control, filter theory and network realization need to check some properties of polynomials, and positivity of polynomials is an important one of such properties [2]. When checking positivity of polynomials using traditional methods for proving inequalities, complexities of algorithms are increasing rapidly as variable number increases [3]. Nowadays, Lu Yang [4] proposed a concise method to prove positivity of homogeneous polynomials (i.e., forms), that is **difference substitution** (DS), or its varied form **weighted difference substitution** (WDS). This method demonstrates great efficiency. [5] further showed that if a form is indeed **positive definite** (PD) or not **positive semidefinite** (PSD), then these properties can be checked by finite steps of WDS. For integral forms, we estimate in this paper upper bounds of step numbers required in checking these properties, they only depend on the variable numbers, the degrees and the upper bounds of absolute values of coefficients of the forms. Therefore, we can also prove whether an integral form is PD through finite steps of WDS.

2 Main Result

We first introduce following definitions and notations according to [5].

Considering $T_n \in \mathbb{R}^{n\times n}$, where

$$T_n = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{n} \\ 0 & \frac{1}{2} & \dots & \frac{1}{n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{n} \end{pmatrix}. \quad (1)$$

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Denote by Θ_n the set of all $n!$ permutations of $\{1,2,\ldots,n\}$, for $(k_1k_2\ldots k_n)\in\Theta_n$, let $P_{(k_1k_2\ldots k_n)}=(a_{ij})_{n\times n}$ be the permutation corresponding $(k_1k_2\ldots k_n)$, that is

$$a_{ij}=\begin{cases}1, & j=k_i\\ 0, & j\neq k_i\end{cases}.$$

Let

$$A_{(k_1k_2\ldots k_n)}=P_{(k_1k_2\ldots k_n)}T_n,$$

and call it the WDS matrix determined by permutation $(k_1k_2...k_n)$, denote by Γ_n the set of all $n!$ such matrices. The variable substitution $\mathbf{x}=A_{(k_1k_2\ldots k_n)}\mathbf{y}$ corresponding $(k_1k_2\ldots k_n)$ is called a WDS, where $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T,\mathbf{y}=(y_1,y_2,\ldots,y_n)^T$, the following set of substitutions

$$\{\mathbf{x}=A_1A_2\cdots A_m\mathbf{y}:A_i\in\Gamma_n\}$$

is called the m-th WDS set.

Suppose $A=(a_{ij})\in\mathbb{R}^{n\times n}$, we call it a normal matrix, and the corresponding substitution a normal substitution if $\sum\limits_{i=1}^n a_{ij}=1, j=1,2,\ldots n$. Thus WDS matrices are normal matrices and WDS substitutions are normal substitutions.

Lemma 2.1. *Let $A=(a_{ij})_{n\times n}=B_1B_2\cdots B_k$, where $B_i(i=1,2,\ldots,k)$ are all normal matrices. Then A is a normal matrix, and for the substitution $\mathbf{x}=A\mathbf{y}$, we have $\sum\limits_{i=1}^n x_i=\sum\limits_{i=1}^n y_i$.*

Proof. Suppose $B_1=(b_{1ij}),B_2=(b_{2ij})$ are normal matrices, let $C=B_1B_2$, and denote by $C=(c_{ij})$, then

$$\sum_{i=1}^n c_{ij}=\sum_{i=1}^n\sum_{k=1}^n b_{1ik}b_{2kj}=\sum_{k=1}^n\left(\sum_{i=1}^n b_{1ik}\right)b_{2kj}=\sum_{k=1}^n b_{2kj}=1.$$

Thus C is normal, and further we can prove A is normal by introduction. Moreover, we have

$$\sum_{i=1}^n x_i=\sum_{i=1}^n\sum_{j=1}^n a_{ij}y_j=\sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}\right)y_j=\sum_{j=1}^n y_j.$$

□

Let $f(\mathbf{x})\in\mathbb{R}[x_1,x_2,\ldots,x_n]$ be a form, we call

$$\mathrm{WDS}(f)=\bigcup_{\theta\in\Theta_n}\{f(A_\theta\mathbf{x})\}\tag{2}$$

the WDS set of f,

$$\mathrm{WDS}^{(m)}(f)=\bigcup_{\theta_m\in\Theta_n}\cdots\bigcup_{\theta_1\in\Theta_n}\{f(A_{\theta_m}\cdots A_{\theta_1}\mathbf{x})\}\tag{3}$$

the m-th WDS set of f for positive integer m, and set $\mathrm{WDS}^{(0)}(f)=\{f\}$.

Denote by $\mathbb{N}$ the set of nonnegative integers, let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$. For a form of degree d

$$f(x_1,x_2,\ldots,x_n)=\sum_{|\alpha|=d}c_\alpha x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},$$

if all coefficients c_α are nonzero, we say f has complete monomials.

Let

$$\mathbb{R}_+^n=\{(x_1,x_2,\ldots,x_n)^T:x_i\geq 0,i=1,\ldots,n\},$$

denote the $(n-1)$-dimensional simplex in $\mathbb{R}^n$ by

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_n)^T : \sum_{i=1}^{n} x_i = 1, (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n \right\}.$$

and let

$$\widetilde{\Delta}_n = \left\{ (x_1, x_2, \ldots, x_n)^T : \sum_{i=1}^{n} x_i \leq 1, (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}_+^n \right\},$$

Definition 2.1. *Suppose $D \subset \mathbb{R}^n$, $f(\mathbf{x}) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, $f(\mathbf{x})$ is PD in D if $f(\mathbf{x}) > 0$ for any $\mathbf{x} \in D \setminus \{\mathbf{0}\}$, and it is PSD in D if $f(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in D \setminus \{\mathbf{0}\}$.*

Obviously, we have,

Lemma 2.2. *A form $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ has the same positivity in $\mathbb{R}_+^n, \Delta_n$ and $\widetilde{\Delta}_n$.*

Denote by $\mathbb{Z}$ the set of integers. We deduce the following result for integral forms.

Theorem 2.1. *Suppose $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is a form of degree d, and the absolute values of its coefficients do not exceed M, then we have*

1. *f is PD in Δ_n, if and only if there exists $m \leq C_p(M, n, d)$, such that each form in $\mathrm{WDS}^{(m)}(f)$ has complete monomials, and its coefficients are all positive;*

2. *f is not PSD in Δ_n (i.e., the minimum of f in Δ_n is negative), if and only if there exists $m \leq C_{nps}(M, n, d)$, such that a form in $\mathrm{WDS}^{(m)}(f)$ has complete monomials, and its coefficients are all negative.*

where

$$C_p(M,n,d) = \left[\frac{\ln\left(2^{d^n} M^{d^n+1} n^{d^{n+1}+d} d^{(n+1)d+nd^n} (d+1)^{(n-1)(n+2)}\right)}{\ln n - \ln(n-1)} \right] + 2 \tag{4}$$

$$C_{nps}(M,n,d) = \left[\frac{\ln\left(2^{d^n+1} M^{d^n+1} n^{d^{n+1}+d} d^{(n+1)d+nd^n} (d+1)^{(n-1)(n+2)}\right)}{\ln n - \ln(n-1)} \right] + 2 \tag{5}$$

Thus, we can completely determine positivity of f through checking positivity of coefficients of forms in $\mathrm{WDS}^{(C_{nps}(M,n,d))}(f)$:

1. If each form in $\mathrm{WDS}^{(C_{nps}(M,n,d))}(f)$ has complete monomials, and its coefficients are all positive, then f is PD in Δ_n;

2. If each form in $\mathrm{WDS}^{(C_{nps}(M,n,d))}(f)$ has a nonnegative coefficient, then f is PSD in Δ_n;

3. If there exists a form in $\mathrm{WDS}^{(C_{nps}(M,n,d))}(f)$ has complete monomials, and its coefficients are all negative, then f is not PSD in Δ_n.

3 Estimate for lower bounds of positive definite integral forms in the simplex

[7] gives estimate for lower bounds of positive definite integral polynomials in simplex, [8] improves the estimate.

Lemma 3.1 ([8]). *Suppose $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ is positive definite in $\widetilde{\Delta}_n$. If the degree of f is d, and absolute values of its coefficients do not exceed M, then*

$$\min_{\widetilde{\Delta}_n} f \geq (2M)^{-d^{n+1}} d^{-(n+1)d^{n+1}}. \tag{6}$$

Indeed, the deduction in [8] has proved the following more general result.

Lemma 3.2 ([8])**.** *Suppose the minimum of $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ in $\widetilde{\Delta}_n$ is not zero. If the degree of f is d, and absolute values of its coefficients do not exceed M, then*

$$\left| \min_{\widetilde{\Delta}_n} f \right| \geq (2M)^{-d^{n+1}} d^{-(n+1)d^{n+1}}. \tag{7}$$

We have the following result for integral forms in Δ_n.

Lemma 3.3. *Suppose the minimum of $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ in Δ_n is not zero. If the degree of f is d, and absolute values of its coefficients do not exceed M, then*

$$\left| \min_{\Delta_n} f \right| \geq C_1(M, n, d). \tag{8}$$

where $C_1(M, n, d) = (2M)^{-d^n} n^{-d^{n+1}-d} d^{-nd^n}$.

Proof. Let $(x_{1,0}, \ldots, x_{n,0})$ be a minimal point of f in Δ_n, then $x_{j,0} \geq \frac{1}{n}, 1 \leq j \leq n$, we can suppose $x_{n,0} \geq \frac{1}{n}$ without loss of generality. Thus

$$\begin{aligned}
\left| \min_{\Delta_n} f \right| &= |f(x_{1,0}, \ldots, x_{n,0})| \\
&= (n x_{n,0})^d \left| f\left(\frac{x_{1,0}}{n x_{n,0}}, \ldots, \frac{x_{n-1,0}}{n x_{n,0}}, \frac{1}{n} \right) \right| \\
&\geq \left| f\left(\frac{x_{1,0}}{n x_{n,0}}, \ldots, \frac{x_{n-1,0}}{n x_{n,0}}, \frac{1}{n} \right) \right|.
\end{aligned}$$

Let

$$\begin{aligned}
g(x_1, \ldots, x_{n-1}) &= n^d f(x_1, \ldots, x_{n-1}, \frac{1}{n}) \\
&= f(n x_1, \ldots, n x_{n-1}, 1),
\end{aligned}$$

then its minimum is not zero in $\widetilde{\Delta}_{n-1}$. Degree of $g \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ is d, and absolute values of its coefficients do not exceed $n^d M$, so from Lemma 3.2, we have

$$\left| \min_{\widetilde{\Delta}_{n-1}} g \right| \geq (2M)^{-d^n} n^{-d^{n+1}} d^{-nd^n}.$$

Since

$$\left(\frac{x_{1,0}}{n x_{n,0}}, \ldots, \frac{x_{n-1,0}}{n x_{n,0}} \right) \in \widetilde{\Delta}_{n-1},$$

we have

$$\begin{aligned}
&\left| f\left(\frac{x_{1,0}}{n x_{n,0}}, \ldots, \frac{x_{n-1,0}}{n x_{n,0}}, \frac{1}{n} \right) \right| \\
\geq & n^{-d} (2M)^{-d^n} n^{-d^{n+1}} d^{-nd^n} \\
= & (2M)^{-d^n} n^{-d^{n+1}-d} d^{-nd^n}.
\end{aligned}$$

Therefore

$$\left| \min_{\Delta_n} f \right| \geq (2M)^{-d^n} n^{-d^{n+1}-d} d^{-nd^n}.$$

∎

4 WDS and barycentric subdivision

In the Δ_n simplex coordinate system, considering a WDS

$$\mathbf{x} = T_n \mathbf{y}, \tag{9}$$

we can see that $\mathbf{a}_1 = (1, 0, ..., 0)^T$ is transformed to $(1, 0, \ldots, 0)^T$, $\mathbf{a}_2 = (\frac{1}{2}, \frac{1}{2}, \ldots, 0)^T$ is transformed to $(0, 1, \ldots, 0)^T$, ..., and $\mathbf{a}_n = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ is transformed to $(0, 0, \ldots, 1)^T$. Moreover, $\mathbf{a}_k (k = 1, 2, \cdots, n)$ is the barycenter of the $(k-1)$-dimensional proper face containing $\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_k$ in Δ_n. Since (9) is a normal substitution, from Lemma 2.1 we know, after transform (9), the corresponding point for any $(x_1, x_2, \cdots, x_n)^T \in \Delta_n$ satisfies $\sum_{i=1}^{n} y_i = 1$, that is, coordinates after transforms are also normal. So, $\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_n$ is a subsimplex of Δ_n after the first barycentric subdivision, it corresponds a WDS matrix $T_n = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n)$.

Similarly, other $n! - 1$ WDS matrices respectively correspond other $n! - 1$ subsimplexes of Δ_n after the first barycentric subdivision. Thus, from geometrical views, a WDS corresponds a barycentric subdivision of Δ_n.

From Lemma 2.1 and the definition of WDS, we know that sequential WDS correspond sequential barycentric subdivisions of Δ_n.

Denote by $\operatorname{diam} \sigma$ the dimension of simplex σ, i.e., maximal distance between vertexes of σ. Comparing with the dimension of original simplex, dimensions of subsimplexes in barycentric subdivision decrease. That is

Lemma 4.1 ([6]). *Let σ be an n-dimensional simplex, σ' is a subsimplex in the barycentric subdivision of σ, then*

$$\operatorname{diam} \sigma' \le \frac{n}{n+1} \operatorname{diam} \sigma. \tag{10}$$

5 Proof of the main result

Proof of the Theorem 2.1. We will prove two propositions respectively.

(I) Sufficiency is obvious. Now we suppose f is positive definite in Δ_n.

Let

$$f(x_1, \ldots, x_n) = \sum_{i_1+\ldots+i_n=d} c_{i_1\cdots i_n} x_1^{i_1} \cdots x_n^{i_n},$$

choose an arbitrary m-th WDS for f

$$\left\{\begin{array}{l} x_1 = \alpha_1 y_1 + (\alpha_1 + \beta_{12}) y_2 + \ldots + (\alpha_1 + \beta_{1n}) y_n \\ x_2 = \alpha_2 y_1 + (\alpha_2 + \beta_{22}) y_2 + \ldots + (\alpha_2 + \beta_{2n}) y_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = \alpha_n y_1 + (\alpha_n + \beta_{n2}) y_2 + \ldots + (\alpha_n + \beta_{nn}) y_n \end{array}\right. , \tag{11}$$

where $\alpha_i, \alpha_i + \beta_{ij} \ge 0, i = 1, \ldots, n, j = 2, \ldots, n$. Since a WDS is a normal substitution,

$$\sum_{i=1}^{n} \alpha_i = 1, \quad \sum_{i=1}^{n} \beta_{ij} = 0, j = 2, \ldots, n. \tag{12}$$

From Lemma 4.1, we have

$$\sqrt{\sum_{i=1}^{n} \beta_{ij}^2} \le \left(\frac{n-1}{n}\right)^m, \quad j = 2, \ldots, n,$$

Further more,

$$|\beta_{ij}| \le \left(\frac{n-1}{n}\right)^m, \quad i = 1, \ldots, n, j = 2, \ldots, n. \tag{13}$$

Let $u=y_1+\ldots+y_n$, then (11) can be written as

$$\begin{cases} x_1=\alpha_1 u+\beta_{12}y_2+\ldots+\beta_{1n}y_n \\ x_2=\alpha_2 u+\beta_{22}y_2+\ldots+\beta_{2n}y_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n=\alpha_n u+\beta_{n2}y_2+\ldots+\beta_{nn}y_n \end{cases}.$$

So

$$\begin{aligned} & f(x_1,\ldots,x_n) \\ =\ & f(\alpha_1 u+\beta_{12}y_2+\ldots+\beta_{1n}y_n,\ldots,\alpha_n u+\beta_{n2}y_2+\ldots+\beta_{nn}y_n) \\ =\ & \sum_{i_1+\ldots+i_n=d} c_{i_1\cdots i_n}(\alpha_1 u+\beta_{12}y_2+\ldots+\beta_{1n}y_n)^{i_1}\cdots(\alpha_n u+\beta_{n2}y_2+\ldots+\beta_{nn}y_n)^{i_n} \\ =\ & \sum_{i_1+\ldots+i_n=d} c_{i_1\cdots i_n}\alpha_1^{i_1}\cdots\alpha_n^{i_n}u^d+\sum_{i_1+\ldots+i_n=d}\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn})y_1^{i_1}\cdots y_n^{i_n} \\ =\ & f(\alpha_1,\ldots,\alpha_n)u^d+\sum_{i_1+\ldots+i_n=d}\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn})y_1^{i_1}\cdots y_n^{i_n} \\ =\ & \sum_{i_1+\ldots+i_n=d}\left(\frac{d!}{i_1!\cdots i_n!}f(\alpha_1,\ldots,\alpha_n)+\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn})\right)y_1^{i_1}\cdots y_n^{i_n} \\ =\ & \sum_{i_1+\ldots+i_n=d}\widetilde{c}_{i_1\cdots i_n}y_1^{i_1}\cdots y_n^{i_n}, \end{aligned} \tag{14}$$

where

$$\widetilde{c}_{i_1\cdots i_n}=\frac{d!}{i_1!\cdots i_n!}f(\alpha_1,\ldots,\alpha_n)+\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn}). \tag{15}$$

and

$$\begin{aligned} & \sum_{i_1+\ldots+i_n=d}\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn})y_1^{i_1}\cdots y_n^{i_n} \\ =\ & \sum_{\substack{i_1+\ldots+i_n=d\\ j_{11}+\ldots+j_{n1}\neq d}} c_{i_1\cdots i_n}\left(\sum_{j_{11}+\ldots+j_{1n}=i_1}\frac{i_1!}{j_{11}!\cdots j_{1n}!}\alpha_1^{j_{11}}\beta_{12}^{j_{12}}\cdots\beta_{1n}^{j_{1n}}u^{j_{11}}y_2^{j_{12}}\cdots y_n^{j_{1n}}\right)\cdots \\ & \left(\sum_{j_{n1}+\ldots+j_{nn}=i_n}\frac{i_n!}{j_{n1}!\cdots j_{nn}!}\alpha_n^{j_{n1}}\beta_{n2}^{j_{n2}}\cdots\beta_{nn}^{j_{nn}}u^{j_{n1}}y_2^{j_{n2}}\cdots y_n^{j_{nn}}\right) \\ =\ & \sum_{\substack{i_1+\ldots+i_n=d\\ j_{11}+\ldots+j_{n1}\neq d}} c_{i_1\cdots i_n}\sum_{\substack{j_{11}+\ldots+j_{1n}=i_1\\ \ldots\ldots\ldots\ldots\\ j_{n1}+\ldots+j_{nn}=i_n}}\frac{\prod_{k=1}^n i_k!}{\prod_{p,q=1}^n j_{pq}!}\prod_{k=1}^n\alpha_k^{j_{k1}}\prod_{p=1}^n\prod_{q=2}^n\beta_{pq}^{j_{pq}}\cdot \\ & u^{j_{11}+\ldots+j_{n1}}y_2^{j_{12}+\ldots+j_{n2}}\cdots y_n^{j_{1n}+\ldots+j_{nn}} \\ =\ & \sum_{\substack{i_1+\ldots+i_n=d\\ j_{11}+\ldots+j_{n1}\neq d}} c_{i_1\cdots i_n}\sum_{\substack{j_{11}+\ldots+j_{1n}=i_1\\ \ldots\ldots\ldots\ldots\\ j_{n1}+\ldots+j_{nn}=i_n}}\frac{\prod_{k=1}^n i_k!}{\prod_{p,q=1}^n j_{pq}!}\prod_{k=1}^n\alpha_k^{j_{k1}}\prod_{p=1}^n\prod_{q=2}^n\beta_{pq}^{j_{pq}}\cdot \\ & \sum_{s_1+\ldots+s_n=j_{11}+\ldots+j_{n1}}\frac{(j_{11}+\ldots+j_{n1})!}{s_1!\cdots s_n!}y_1^{s_1}y_2^{j_{12}+\ldots+j_{n2}+s_2}\cdots y_n^{j_{1n}+\ldots+j_{nn}+s_n}, \end{aligned}$$

from (12) and (13), we have

$$\begin{aligned}
&\left| c_{i_1\cdots i_n}\frac{\prod_{k=1}^n i_k!}{\prod_{p,q=1}^n j_{pq}!}\prod_{k=1}^n \alpha_k^{j_{k1}}\prod_{p=1}^n\prod_{q=2}^n \beta_{pq}^{j_{pq}}\frac{(j_{11}+\ldots+j_{n1})!}{s_1!\cdots s_n!}\right| \\
\leq & M(d!)^{n+1}\left(\frac{n-1}{n}\right)^{m\sum_{p=1}^n\sum_{q=2}^n j_{pq}} \\
= & M(d!)^{n+1}\left(\frac{n-1}{n}\right)^{m(d-\sum_{p=1}^n j_{p1})} \\
\leq & M(d!)^{n+1}\left(\frac{n-1}{n}\right)^{m} \\
\leq & Md^{(n+1)d}\left(\frac{n-1}{n}\right)^{m}.
\end{aligned}$$

There are

$$\binom{d+n-1}{n-1}\leq (d+1)^{n-1}$$

nonnegative integer tuples $(i_1,\ldots,i_n)$ satisfying $i_1+\ldots+i_n=d$, so $\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn})$ is summed up by terms whose number does not exceed

$$(d+1)^{(n-1)(n+2)},$$

and the absolute value of each term does not exceed

$$Md^{(n+1)d}\left(\frac{n-1}{n}\right)^{m}.$$

Therefore

$$|\phi_{i_1\cdots i_n}(\alpha_1,\ldots,\beta_{nn}))|\leq Md^{(n+1)d}(d+1)^{(n-1)(n+2)}\left(\frac{n-1}{n}\right)^{m}. \tag{16}$$

From Lemma 3.3, we have

$$\frac{d!}{i_1!\cdots i_n!}f(\alpha_1,\ldots,\alpha_n)\geq C_1(M,n,d). \tag{17}$$

From (15), (16), (17), we know that in order that $\widetilde{c}_{ijk}>0$, it suffices

$$C_1(M,n,d)>Md^{(n+1)d}(d+1)^{(n-1)(n+2)}\left(\frac{n-1}{n}\right)^{m}.$$

That is

$$m>\frac{\ln\left(2^{d^n}M^{d^n+1}n^{d^{n+1}+d}d^{(n+1)d+nd^n}(d+1)^{(n-1)(n+2)}\right)}{\ln n-\ln(n-1)}. \tag{18}$$

(II) Sufficiency is also obvious. Now we suppose the minimum of f in Δ_n is negative, and a minimal point is $(a_1,\ldots,a_n)$. From Lemma 3.3, we have

$$|f(a_1,\ldots,a_n)|\geq C_1(M,n,d).$$

Suppose $(y_{11},\ldots,y_{1n})^T,(y_{21},\ldots,y_{2n})^T\in\Delta_n$, $(x_{11},\ldots,x_{1n})^T,(x_{21},\ldots,x_{2n})^T$ are coordinates satisfying (11), from Lemma 2.1, we have $(x_{11},\ldots,x_{1n})^T,(x_{21},\ldots,x_{2n})^T\in\Delta_n$. From the correspondence of WDS and barycentric subdivisions, we have

$$\sqrt{(x_{11}-x_{21})^2+\ldots+(x_{1n}-x_{2n})^2}\leq\left(\frac{n-1}{n}\right)^{m}.$$

Let $\delta_j = x_{1j} - x_{2j}, j = 1, \ldots, n$, then there exists $\gamma \in (0,1)$, such that

$$
\begin{aligned}
&|f(x_{11}, \ldots, x_{1n}) - f(x_{21}, \ldots, x_{2n})| \\
=&\left| \left(\delta_1 \frac{\partial}{\partial x_1} + \ldots + \delta_n \frac{\partial}{\partial x_n} \right) f(x_{21} + \gamma\delta_1, \ldots, x_{2n} + \gamma\delta_n) \right| \\
\leq &\sum_{j=1}^{n} |\delta_j| \left| \frac{\partial}{\partial x_j} f(x_{21} + \gamma\delta_1, \ldots, x_{2n} + \gamma\delta_n) \right| \\
\leq &\left(\frac{n-1}{n} \right)^m \sum_{j=1}^{n} \sum_{i_1+\ldots+i_n=d} \left| c_{i_1 \cdots i_n} (x_{21} + \gamma\delta_1)^{i_1} \cdots (x_{2j} + \gamma\delta_j)^{i_j - 1} \cdots (x_{2n} + \gamma\delta_n)^{i_n} \right| \\
\leq &\left(\frac{n-1}{n} \right)^m \sum_{j=1}^{n} \sum_{i_1+\ldots+i_n=d} M 2^{d-1} \\
=&nM2^{d-1} \binom{d+n-1}{n-1} \left(\frac{n-1}{n} \right)^m \\
\leq &nM2^{d-1}(d+1)^{n-1} \left(\frac{n-1}{n} \right)^m .
\end{aligned} \tag{19}
$$

Thus if m is sufficiently large, such that

$$
nM2^{d-1}(d+1)^{n-1} \left(\frac{n-1}{n} \right)^m \leq \frac{1}{2} C_1(M, n, d),
$$

i.e.,

$$
m \geq \frac{\ln \left(2^{d^n+d} M^{d^n+1} n^{d^{n+1}+d+1} d^{nd^n} (d+1)^{n-1} \right)}{\ln n - \ln(n-1)}, \tag{20}
$$

m-th WDS (11) satisfies

$$
f(\alpha_1, \ldots, \alpha_n) \geq \frac{1}{2} C_1(M, n, d). \tag{21}
$$

From the deduction of (I), we can see that if

$$
m > \frac{\ln \left(2^{d^n+1} M^{d^n+1} n^{d^{n+1}+d} d^{(n+1)d+nd^n} (d+1)^{(n-1)(n+2)} \right)}{\ln n - \ln(n-1)}, \tag{22}
$$

and m satisfies (20), there exists a form in $\mathrm{WDS}^{(m)}(f)$, it has complete monomials, and all coefficients are negative. Comparing the right hand sides of (20) and (22), the latter is larger, so it suffices (22).

□

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